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COMPLETE STATISTICAL RANKING OF POPULATIONS, WITH TABLES AND APPLICATIONS

by

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ABSTRACT

Recently a new statistical methodology, developed over the last 3 decades, has become available to practitioners. This methodology is called "ranking and selection" theory. In this article we review procedures for completely ranking a set of populations (from "best", "second best", etc., down to "worst"), give new tables needed to implement these procedures, and consider several practical examples using real data.

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1. INTRODUCTION

Since 1954 (Bechhofer (1954)) a new statistical methodology has been developing. This methodology is called "ranking and selection" theory, and has recently become accessible to practitioners (Gibbons, Olkin, and Sobel (1977), Dudewicz (1980)). In this section we review the problem of completely ranking a set of populations (from "best", "second best", etc., down to "worst"). New tables needed to implement these procedures are given in Section 5, with a discussion of their construction in Section 3. Practical examples using real data are analyzed in Section 4. These examples should aid researchers in many fields in proper use of this new methodology.

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2. PROCEDURE

IZ FORMULATION

Let $\pi_1,\pi_2,\ldots,\pi_k$ represent $k(\geq 2)$ independent sources of random variables. Assume observations from π_i are normally distributed $N(\mu_i,\sigma_i^2)$ $(1 \leq i \leq k)$ with means $\mu_1,\ldots,\mu_k$ and variances $\sigma_1^2,\ldots,\sigma_k^2$ all unknown. Assume (*Indifference Zone* Formulation) that the goal is to completely rank the populations from that with the largest mean $\mu_{[k]}$ down through that with the smallest mean $\mu_{[1]}$ (where $\mu_{[1]} \leq \cdots \leq \mu_{[k]}$ denote the ordered means) in such a way that the probability of correct ranking P(CR) satisfies

$$P(CR) \geq P^* \text{ whenever } \mu_{[j]} - \mu_{[j-1]} \geq \delta_j^* \quad (j=2,\ldots,k) \tag{1}$$

where P^* and $\delta_2^*,\ldots,\delta_k^*$ $(1/k! < P^* < 1; 0 < \delta_2^*,\ldots,\delta_k^*)$ are specified in advance by the experimenter. It has been shown by Dudewicz and van der Meulen (1980) that this is guaranteed by the following procedure $\mathcal{P}_{DD}$(CR-IZ).

<u>PROCEDURE</u> $\mathcal{P}_{DD}$(CR-IZ). Take an initial sample $X_{i1},\ldots,X_{in_0}$ of size $n_0(\geq 2)$ from π_i and define

$$\bar{X}_i(n_0) = \sum_{j=1}^{n_0} X_{ij}/n_0 \ , \ s_i^2 = \sum_{j=1}^{n_0} (X_{ij} - \bar{X}_i(n_0))^2/(n_0 - 1), \tag{2}$$

$$n_i = \max\{n_0 + 1, [(s_i h/\delta^*)^2]\} \tag{3}$$

where $\delta^* > 0$ is arbitrary and $h > 0$ is the unique solution of the equation

$$P^* = P[Y_1 < Y_2 + \delta_2^* h/\delta^*, Y_2 < Y_3 + \delta_3^* h/\delta^*, \ldots, Y_{k-1} < Y_k + \delta_k^* h/\delta^*] \tag{4}$$

where $Y_1, \ldots, Y_k$ are independent Student's - t random variables each with $n_0 - 1$ degrees of freedom. (Notice that

$$P[Y_1 < Y_2 + \delta_2^* h/\delta^*, Y_2 < Y_3 + \delta_3^* h/\delta^*, Y_3 < Y_4 + \delta_4^* h/\delta^*, \ldots, Y_{k-1} < Y_k + \delta_k^* h/\delta^*]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{y_k + \delta_k^* h/\delta^*} \cdots \int_{-\infty}^{y_4 + \delta_4^* h/\delta^*} \int_{-\infty}^{y_3 + \delta_3^* h/\delta^*}$$

$$F_{n_0}(y_2 + \delta_2^* h/\delta^*) f_{n_0}(y_2) f_{n_0}(y_3) \cdots f_{n_0}(y_{k-1}) f_{n_0}(y_k) dy_2 dy_3 \cdots dy_{k-1} dy_k \tag{5}$$

where $F_{n_0}(\cdot)$ and $f_{n_0}(\cdot)$ are respectively the distribution and density function of a Student's - t random variable with n_0-1 degrees of freedom.) In (3), $[y]$ denotes the smallest integer $\geq y$. Take $n_i - n_0$ additional observations $X_{i,n_0+1}, \ldots, X_{in_i}$ from π_i and define

$$\tilde{\bar{X}}_i = \sum_{j=1}^{n_i} a_{ij} X_{ij} \tag{6}$$

($i = 1, \ldots, k$). (Here the a_{ij}'s, $j = 1, \ldots, n_i$, $1 \leq i \leq k$, are chosen so that

$$\sum_{j=1}^{n_i} a_{ij} = 1, \; a_{i1} = \ldots = a_{in_0}, \text{ and } s_i^2 \sum_{j=1}^{n_i} a_{ij}^2 = (\delta^*/h)^2. \tag{7}$$

To be specific, take the positive radical solution from $a_{i,n_0+1} = \ldots = a_{in_i}$.) Finally, assert (for $i = 1, \ldots, k$) that the population which yielded $\tilde{\bar{X}}_{[i]}$ has mean $\mu_{[i]}$ (where $\tilde{\bar{X}}_{[1]} \leq \ldots \leq \tilde{\bar{X}}_{[k]}$ denote $\tilde{\bar{X}}_1, \ldots, \tilde{\bar{X}}_k$ in numerical order).

If one's indifference zones are fully symmetric in that $\delta_2^* = \ldots = \delta_k^* = \delta^*$ (say), then the $h = h_{n_0,k}(P^*)$ to be used is the unique solution of

$$P^* = P[Y_1 < Y_2 + h, Y_2 < Y_3 + h, Y_3 < Y_4 + h, \ldots, Y_{k-1} < Y_k + h] \qquad (8)$$

where $Y_1, \ldots, Y_k$ are independent Student's - t random variables each with $n_0 - 1$ degrees of freedom. In what follows, we suppose $\mathcal{P}_{DD}$(CR-IZ) is always used with $\delta_2^* = \ldots = \delta_k^* = \delta^*$; the corresponding h, which is denoted by

$$h = h_{n_0,k}(P^*, \delta_2^*/\delta^*, \ldots, \delta_k^*/\delta^*) \qquad (9)$$

where $\delta_i^*/\delta^* = 1$ $(i = 2, \ldots, k)$, is tabulated in Section 5.

PPF FORMULATION

In the setting considered above, another requirement is often desired by experimenters : that they be 100P* % sure the ranking they ultimately state is correct up to interchanges of populations whose true means differ by δ^* or less. Thus, the *Preferred Population* Formulation (PPF) of the complete ranking problem has, as its goal, statement of a complete ranking of the populations in such a way that

$$P(CR) \geq P^* \qquad (10)$$

for all possible parameter configurations, where P^* $(1/k! < P^* < 1)$ is specified in advance by the experimenter and event "CR" is considered to occur if the order specified is correct or can be made correct by one or more interchanges of assertions involving populations whose true means differ by at most δ^* $(\delta^* > 0)$.

Now it is shown in Dudewicz and van der Meulen (1980) that procedure $\mathcal{T}_{DD}(CR - IZ)$ with $\delta_2^* = \ldots = \delta_k^* = \delta^*$ given above satisfies $P(CR) \geq P^*$ with this new definition of event CR. Thus the following procedure achieves the goal.

PROCEDURE $\mathcal{T}_{DD}(CR - PPF)$. Choose $\delta_2^* = \ldots = \delta_k^* = \delta^*$ and proceed according to procedure $\mathcal{T}_{DD}(CR-IZ)$.

3. CONSTRUCTION OF TABLES FOR $\mathcal{T}_{DD}(CR-IZ)$ AND $\mathcal{T}_{DD}(CR-PPF)$

In order to tabulate h which solves equation (8), for each n_0 (10,15,20,25,30) of interest we called a Monte Carlo evaluation routine SIMUL, given below in Figure 1, in a pinch process attempting to converge on the root.

For computational efficiency, subroutine SIMUL does evaluation of the P(CR) simultaneously for each of $k = 2,3,\ldots,25$ and also simultaneously at 144 h values, with each h value chosen so as to be closest to the root of the corresponding (8) when $P^* = .75,.80,.85,.90,.95,.975$ and $k = 2,3,\ldots,25$. SIMUL looks at 10,000 samples of k Student's - t random variables $Y_1,\ldots,Y_k$ with $n_0 - 1$ degrees of freedom. These are constructed from kn_0 independent standard normal random variables generated using the Box-Muller transformation to normality on pseudo-random numbers from generator UNI with seeds IX = 524,287 and JX = 654,345,465; this generator, which has a period of $2^{46} - 2^{29} \approx 7 \times 10^{13}$, has been found to have good properties in extensive testing by Dudewicz and Ralley (1981). Then SIMUL reports the proportion of the 10,000 samples where a completely correct ranking was achieved, in the sense that

$$Y_1 < Y_2 + h < Y_3 + 2h < \ldots < Y_{k-1} + (k-2)h < Y_k + (k-1)h. \qquad (11)$$

These computer runs were preceded by a Monte Carlo tabulation of P(CR) as a function of $h = 0.0\ (0.1)\ 5.1$, from which intervals in which the true root of (8) lies were determined for use in the later pinch process just described, and were followed by supplemental runs where needed (for additional accuracy, or where the preliminary interval used did not in fact contain the true root). These runs used random number generator RANDOM with seed INT = 524,287. This generator was recommended by Dudewicz (1976) after preliminary investigations for goodness and speed. The results of the pinch process are given in the tables of Section 5, where for each value of k, a separate table is given containing as entries the h values corresponding to a fixed choice of P^* and n_0. E.g. for $k = 10$, $P^* = 0.975$, and $n_0 = 20$, one finds $h = 4.29$.

The tables also include values of h for the case $n_0 = \infty$ (used when one knows the variances of the populations), calculated by a Monte Carlo simulation procedure similar to the one described above but with use of standard normal random variables instead of Student's - t random variables. In the latter case the pseudo-random numbers were generated using generator RANDOM. These values also facilitate interpolation for $n_0 > 30$.

The tables also list values of h for the cases $n_0 = 2(1)9$. As values of $n_0 \geq 10$ are recommended for use in usual practice, fewer decimals are reported in the tables for $n_0 < 10$. Fewer decimals were also reported for $P^* = .99$, but there because (see the accuracy analysis below) further decimals would be inaccurate. These table parts were obtained from the Monte Carlo tabulation of P(CR) as a function of $h = 0.0(0.1)5.1$ (supplemented by additional evaluations for cases where $h > 5.1$).

```
      SUBROUTINE SIMUL(HM,PCR,NO)
      DIMENSION Y(25),U(750),X(750),NCR(24,6),PCR(24,6),HM(24,6)
      IX=524287
      JX=654345465
      CALL RSTART(IX,JX)
      DO 2 L1=1,24
      DO 2 L2=1,6
    2 NCR(L1,L2)=0
      N=0
      N1=((NO*25+1)/2)*2
    3 DO 50 M1=1,N1
   50 U(M1)=UNI(0)
      N2=(NO*25)/2
      DO 51 M2=1,N2
      M2A=2*M2-1
      M2B=M2A+1
      X(M2A)=SQRT(-2.*ALOG(U(M2A)))*COS(2.*3.141593*U(M2B))
   51 X(M2B)=SQRT(-2.*ALOG(U(M2A)))*SIN(2.*3.141593*U(M2B))
      IF (N2.LT.((NO*25.)/2.)) X(N1-1)=SQRT(-2.*ALOG(U(N1-1)))*
     +COS(2.*3.141593*U(N1))
      N3=NO-1
      DO 52 J3=1,25
      SUM=0
      DO 53 J4=1,N3
   53 SUM=SUM+X((J3-1)*NO+1+J4)**2
   52 Y(J3)=X((J3-1)*NO+1)/SQRT(SUM/N3)
      DO 5 I1=1,24
      DO 5 I2=1,6
      H=HM(I1,I2)
      K=1
    6 IF ((Y(K)+(K-1.)*H).GE.(Y(K+1)+K*H)) GOTO 5
      K=K+1
      IF (K.LE.I1) GOTO 6
      NCR(I1,I2)=NCR(I1,I2)+1
    5 CONTINUE
      N=N+1
      IF (N.LT.10000) GOTO 3
      DO 7 L3=1,24
      DO 7 L4=1,6
    7 PCR(L3,L4)=NCR(L3,L4)/10000.
      RETURN
      END
```

Figure 1*. Subroutine SIMUL

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* Thanks are due to Mr. P. Darius for preparing the printout used in Figure 1.

ACCURACY ANALYSIS

A number of checks of accuracy of the computations were carried out, and will now be described briefly. A P(CR) computation using the procedure described above is accurate (with 95 % probability) to within $\pm 2\sqrt{P^*(1-P^*)/10{,}000}$, tabulated in Table 1.

Table 1.

P^*	$2\sqrt{P^*(1-P^*)/10{,}000}$
.50	.010
.75	.008
.80	.008
.85	.007
.90	.006
.95	.004
.975	.003
.99	.002

Now, as a first check, when $h = 0.0$ we know theoretically (from (8)) that $P(CR) = 1/k!$, which was (within the Table 1 accuracies of the Monte Carlo) confirmed in our tabulation of P(CR) as a function of $h = 0.0(0.1)$ 5.1.(E.g., when $k=2$ and $n_0=25$ we found $P(CR) = .4967$, which is within .01 of the true value .5000.) As a second check, when $k=2$ the problem of a complete ranking is equivalent to the problem of selecting the best (since then the population not selected must be inferior if the selection of the best has been correctly made). For this latter problem, Dudewicz, Ramberg, and Chen (1975) have tabulated h in their Table 4. Comparison shows our method leads to values correct to $\pm$ 2 units in the last place shown (84 entries), $\pm$ 3 units (4 entries), $\pm$ 4 units (1 entry), $\pm$ 5 units (1 entry). In addition 1 entry matched exactly.

Also for k=2, Bechhofer (1954) has tabulated h in his table I for the case $n_0 = \infty$. Again comparison shows our method leads to values correct to $\pm$ 1 unit in the last place shown (2 entries), $\pm$ 2 units (2 entries), $\pm$ 3 units (1 entry).(Table I of Bechhofer (1954) does not have entries for $P^* = .85, .975$.)

Finally, our tables should be monotone increasing in k and P^*, and monotone decreasing in n_0. This is fulfilled except for the monotonicity in n_0 where in one case $h_{20,k}(P^*) < h_{25,k}(P^*)$ and in 5 cases $h_{25,k}(P^*) < h_{30,k}(P^*)$. This however can be explained from the accuracy analysis and the flatness of the curve $h(n_0)$ near larger values of n_0.

To illustrate the accuracy of the tables as presented, let us consider the entry h = 2.06 when k = 4, n_0 = 15, P^* = .75. Here the pinch process yielded the results :

Table 2.

h	P(CR) Estimated
2.05000	.7473
2.05625	.7485
2.06250	.7518
2.07500	.7551
2.10000	.7610

(though not in this order). As the estimated P(CR) is good (probability

95 %) to within $\pm$ .008 when $P^* = .75$, we have established that the true value h is less than 2.10... but cannot distinguish 2.06 from 2.07, etc. (see the graph in Figure 2 below). Thus (on the upper side) 2.06 is within 4 units of the true root h (which cannot exceed or even equal 2.10). If the estimated P(CR) at the h value 2.07 had been greater than .758, we would then have attempted to distinguish another significant digit. This analysis was carried through for the $k = 10$ and $k = 25$ tables, as well as the $k = 2$ table (where - see above - independent confirmation is available from Dudewicz, Ramberg, and Chen (1975)). It indicated an absolute error in tabled values of usually ≤ 5 or 6 units in the last place reported, when $n_0 \geq 3$. (At $n_0 = 2$ the entries can possibly be subject to much greater errors, but this is not of substantial interest as $n_0 \geq 10$ is the usual case for practical use.) As our results for the case $k = 2$ were in fact much better than these rough bounds would suggest, we believe these tables are of such accuracy as can be safely used in practice, and do not expect inaccuracy to exceed 5 units in the last place reported.

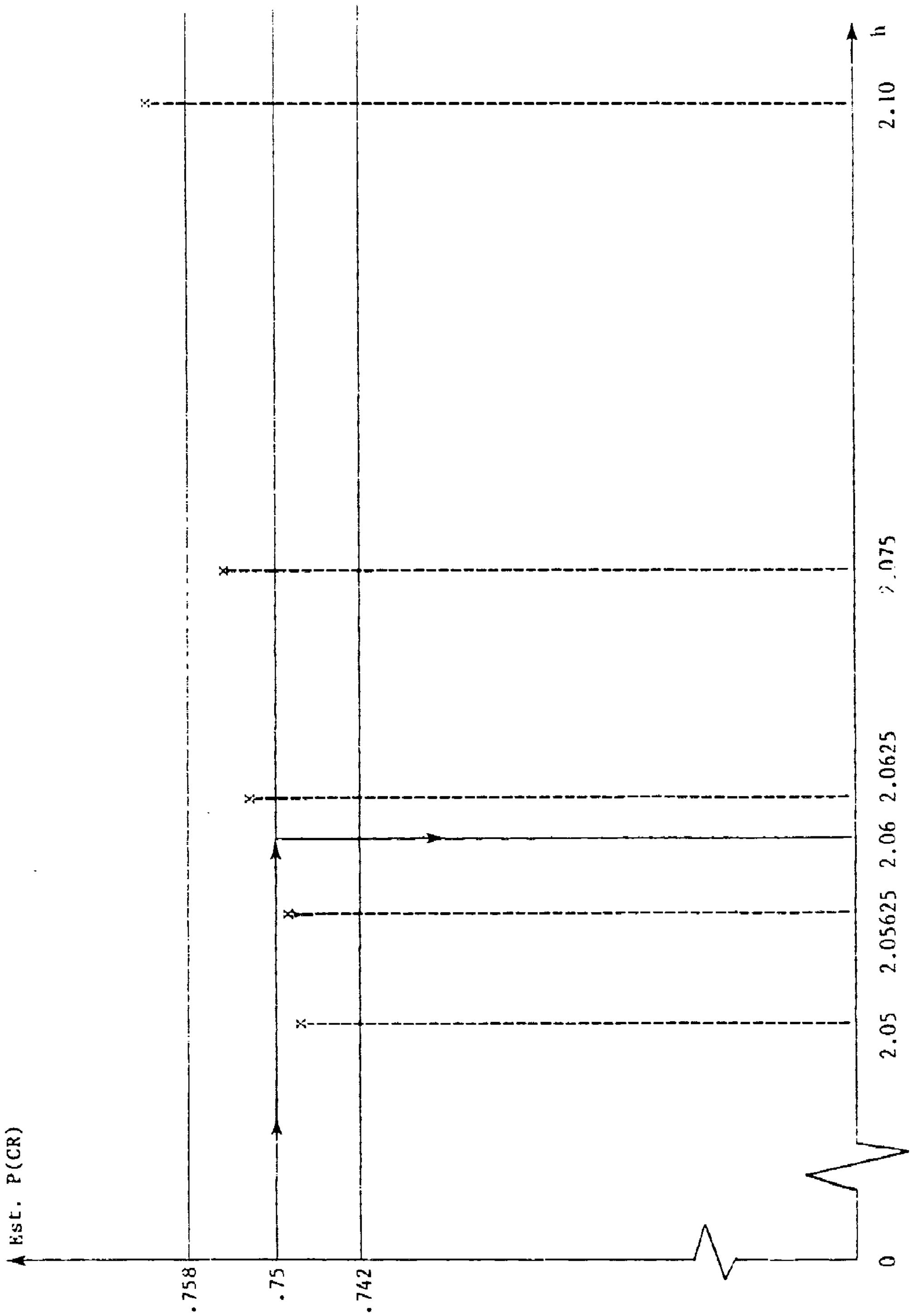

Figure 2. Illustration of Accuracy Analysis

4. EXAMPLES

Example 1. Shuttle-Run Timings. Data on times in the so-called shuttle-run (480 meters) of 62 girls aged (by rounded off Larson age*, as given by Larson (1974)) 7, 8, and 9 are given below. (Thanks for this dataset are due to Mrs. M. Vuylsteke-Wauters, co-promotor at K.U. Leuven of E. Anthonissen (1981).) Here $k = 3$ populations are present with respectively 27, 17 and 18 observations. Let us choose $n_0 = 10$ observations in our first stage. If we desire to have probability $P^* = .95$ that our ultimate ranking is PPF-correct, we will (from our tables) need to use $h = 3.19$.

From the data we find $\bar{X}_1(10) = 316.80$, $\bar{X}_2(10) = 280.40$, $\bar{X}_3(10) = 275.40$, $s_1 = 28.69$, $s_2 = 23.00$, $s_3 = 22.35$. If we wish to be 95 % ($100 \times P^*$%) sure our ultimate ranking is correct up to interchanges involving populations whose true means differ by at most $\delta^*(\delta^* > 0)$, we will need total sample sizes of

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* Larson age is defined as :

date of examination (decimals) - date of birth (decimals).

(Three decimals are reported, being calculated from

$$\left(\frac{(\text{number of the day in the year}) - 1}{365}\right).)$$

For example, date of examination : 11th January 1981 = 81.027

date of birth : 20th July 1973 = 73.548

age at time of examination = 7.479 years

Here "age 7" means a Larson age between 6.95 and 7.95, i.e. an age between 6 years 11 2/5 months and 7 years 11 2/5 months.

$$n_1 = \left[\frac{(3.19)^2(28.69)^2}{(\delta^*)^2}\right], \quad n_2 = \left[\frac{(3.19)^2(23.00)^2}{(\delta^*)^2}\right], \quad n_3 = \left[\frac{(3.19)^2(22.35)^2}{(\delta^*)^2}\right] \tag{12}$$

from the three respective populations. Since we are constrained to $n_1 \leq 27$, $n_2 \leq 17$, $n_3 \leq 18$, we choose δ^* as small as possible (the strongest resulting guarantee) without violating these constraints. Thus, from (3) we find $\delta^* = 17.80$ seconds, which yields $n_1 = 27$, $n_2 = 17$, $n_3 = 17$.

Solving for the a_{ij}'s in display (6) we find for population 1 (7 year olds), taking $a_1 \equiv a_{1,1} = a_{1,2} = \ldots = a_{1,10}$ and $b_1 \equiv a_{1,11} = \ldots = a_{1,27}$, that a_1 and b_1 solve the system

$$\begin{cases} (28.69)^2(10\ a_1^2 + 17\ b_1^2) = (17.80)^2/(3.19)^2 \\ 10\ a_1 + 17\ b_1 = 1. \end{cases} \tag{13}$$

Taking the positive radical solution yields $a_1 = .0230$ and $b_1 = .0412$. Hence

$$\tilde{\bar{X}}_1 = (.0230)(3168) + (.0412)(260 + 311 + \ldots + 311) = 292.97. \tag{14}$$

Similarly for population 2 (8 year olds) $a_2 \equiv a_{2,1} = \ldots = a_{2,10} = .0576$, $b_2 \equiv a_{2,11} = \ldots = a_{2,17} = .0605$, $\tilde{\bar{X}}_2 = 282.29$. Finally, for population 3 (9 year olds) $a_3 \equiv a_{3,1} = \ldots = a_{3,10} = .0468$, $b_3 \equiv b_{3,11} = \ldots = b_{3,17} = .0760$, and $\tilde{\bar{X}}_3 = 269.49$.

We conclude that (in increasing order of speed) the groups are :

Fastest	= 9 year olds
2^{nd} Fastest	= 8 year olds
Slowest	= 7 year olds

Table 3. Girls' Shuttle-Run Times (Seconds)

	Age (Rounded off Larson Years)		
N°	7	8	9
1	349	264	286
2	352	257	239
3	347	284	275
4	313	298	253
5	290	284	256
6	288	315	293
7	298	262	283
8	283	254	314
9	347	270	265
10	301	316	230
11	260	291	239
12	311	328	277
13	291	282	236
14	256	261	260
15	247	298	318
16	291	286	257
17	273	248	263
18	269		284
19	278		
20	267		
21	286		
22	300		
23	300		
24	280		
25	276		
26	312		
27	311		

with 95 % confidence, subject to reversals on groups with true means closer than $\delta^* = 17.80$.

Example 2. Agricultural Treatments and Fertilizers. Each year agricultural experiments are run at Heverlee and other locations (stations) in Belgium so as to be able to advise farmers in the region of optimal varieties of agricultural products.

In the 1979/80 season at the "Veredelingsstation Heverlee" experiments were run on 3 varieties of Winterwheat with two sets of treatments (with fertilizers and without fertilizers *). Thus we have two sets of $k = 3$ populations each of which we wish to rank separately on yield from best, second best, to worst. The yields are put on a kg/m^2 basis as plots are of varying sizes, i.e. either 15.12 or 11.34 m^2 (but each plot received 300 seeds/m^2). The data obtained are given in Table 4 below. (Thanks for this dataset are due to ir. J. Niclaes and Dr. L. Kempeneers of the Veredelingsstation.)

With the choice of $n_0 = 15$ observations, we are interested in the following questions.

1. If $\delta^* = .03$ kg/m^2 is a difference of basic interest (while interchanges in ordening of populations closer than δ^* in their means are not of strong importance), with $n_0 = 15$ what P^* can we guarantee of a fully correct ordering in the PPF formulation for non-fertilized treatments ? For fertilized treatments ?

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* The fertilizers which were used are : 100 units/ha Nitrogen, 1.5 l/ha Chloormequatchloride, Herbicides (Metabenzthiazuron 3 kg/ha and Mecoprop + Ioxynil 4 l/ha), and Fungicides (4 kg/ha Spuitzwavel, 0.4 kg/ha Benlate, 3 kg/ha Thiofanaat-methyl +maneb, and again 2.5 kg/ha Spuitzwavel).

Table 4. Agricultural Yields (kg/m^2), at two substations of the "Veredelings-station Heverlee", 1980

Plot Number	Variety 1		Variety 2		Variety 3	
	W/O Fert.	Fert.	W/O Fert.	Fert.	W/O Fert.	Fert.
1	.562*	.754	.575*	.780	.602*	.827
2	.648*	.794	.609*	.800	.589*	.813
3	.549*	.787	.615*	.800	.542*	.774
4	.516*	.708	.516*	.800	.595*	.767
5	.503	.721	.485	.761	.467	.787
6	.538	.734	.564	.721	.512	.708
7	.476	.747	.538	.728	.512	.747
8	.467	.741	.450	.714	.485	.622
9	.503	.701	.582	.721	.520	.734
10	.582	.694	.494	.681	.503	.668
11	.485	.741	.529	.708	.538	.728
12	.538	.721	.600	.668	.529	.694
13	.467	.761	.406	.714	.467	.747
14	.511	.708	.520	.761	.414	.767
15	.503	.787	.582	.728	.547	.761
16	.467	1.058	.600	-	.458	.728
$\bar{x}(15)$	.523	.740	.538	.739	.521	.743
s	.051	.030	.057	.043	.056	.053

In order to take only one more observation in each group, it follows from (3) that we have to look for the largest possible h value such that

$$\left(\max_{i=1,2,3} s_i^2\right) h^2/\delta^{*2} \leq 16. \qquad (15)$$

In the case of treatments without fertilizer this means that $h^2 = (.03)^2\ 16/(.057)^2$. Then from the Monte Carlo tabulation of P(CR) as a function of h (which was obtained in the course of constructing the tables

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* Data accompanied with * are obtained from a 11.34 m^2 plot. All other data come from an 15.12 m^2 plot.

of Section 5) we found that $P^* = .8340$ can be guaranteed of a fully correct ordering in the PPF formulation. Similarly, in the case of treatments with fertilizer $P^* = .8531$ can be guaranteed, assuming 16 observations for each population are available.

Moreover, in cases of no fertilizer, analysis based on the procedure described in Section 2 implies that one can be 83.40 % sure that a ranking where variety 2 gives most yield, variety 1 second most, and variety 3 least, is correct up to interchanges of the varieties whose true means differ by $\delta^* = 300$ kg/ha $= .03$ kg/m^2 or less. In the cases of fertilizer no such conclusion can be made because of the missing 16th observation for variety 2 (though a similar analysis could be made using $n_0 = 14$).

2. With $P^* = .95$ and $n_0 = 15$, what δ^* can be guaranteed for the PPF formulation for non-fertilized treatments ? For fertilized treatments ?

From (3) again, we have to look for the smallest possible δ^* value such that

$$\left(\max_{i=1,2,3} s_i^2\right) (2.98)^2/\delta^{*2} \leq 16 \tag{16}$$

where the h value 2.98 follows from the tables. For the case of no fertilizer this gives $\delta^* = .0425$, while in the case of treatment with fertilizer $\delta^* = .0395$ can be guaranteed.

3. If $\delta^* = .03$ kg/m^2 and $P^* = .95$, which number of observations, additional to $n_0 = 15$,should be used in future trials so as to be able to make appropriate guarantees on ordering analysis in the PPF formulation ?

To find the required number of additional observations, the value

$$\left[\frac{\left(\max\limits_{i=1,2,3} s_i^2\right)(2.98)^2}{(.03)^2}\right] \qquad (17)$$

is to be calculated. This gives in the no-fertilizer treatment case 33-15 = 18 additional observations, and for the data with fertilizing treatment 28-15 = 13 additional observations. Thus, in future trials the experimenter is recommended to design the experiment with a least 33 plots per variety in the fertilizer treatment case (28 in the no-fertilizer treatment case) when δ^* is chosen .03 kg/m^2 and P^* is to be .95.

5. TABLES NEEDED FOR IMPLEMENTATION

Table 5. The Solution h of Equation (8) for : $k = 2(1)25$; $P^* = .75(.05).95, .975, .99$; $n_0 = 2(1)10(5)30,\infty$

k = 2

P^* \ n_0	2	3	4	5	6	7	8	9	10	15	20	25	30	∞
.75	2.0	1.4	1.2	1.1	1.1	1.1	1.0	1.0	1.04	1.01	0.98	0.98	0.96	0.96
.80	2.8	1.8	1.6	1.4	1.4	1.4	1.3	1.3	1.30	1.26	1.23	1.23	1.20	1.18
.85	4.0	2.3	1.9	1.8	1.8	1.7	1.6	1.6	1.60	1.53	1.52	1.51	1.49	1.46
.90	6.0	3.2	2.5	2.3	2.2	2.1	2.0	2.0	1.96	1.90	1.92	1.87	1.85	1.80
.95	12	4.7	3.5	3.2	2.9	2.8	2.8	2.6	2.60	2.50	2.48	2.40	2.38	2.35
.975	24	6.7	4.6	4.0	3.6	3.5	3.4	3.2	3.16	2.97	2.97	2.90	2.83	2.83
.99	62	10	6	5	4.4	4.4	4.3	4.0	3.6	3.5	3.5	3.5	3.3	3.3

k = 3

P^* \ n_0	2	3	4	5	6	7	8	9	10	15	20	25	30	∞
.75	4.7	2.6	2.2	2.0	1.9	1.9	1.8	1.8	1.76	1.71	1.70	1.68	1.67	1.61
.80	5.9	3.1	2.5	2.3	2.2	2.1	2.0	2.0	1.97	1.92	1.91	1.88	1.86	1.81
.85	8.1	3.7	2.9	2.7	2.5	2.4	2.3	2.3	2.23	2.18	2.16	2.12	2.11	2.03
.90	12.1	4.6	3.5	3.1	2.9	2.8	2.7	2.6	2.61	2.50	2.47	2.41	2.42	2.32
.95	23	6.7	4.5	4.0	3.6	3.5	3.3	3.2	3.19	2.98	2.95	2.91	2.89	2.80
.975	47	9	5.8	4.9	4.3	4.1	4.0	3.8	3.70	3.47	3.41	3.4	3.33	3.21
.99	120	15	8	7	6	4.9	4.8	4.5	4.3	4.0	3.9	3.9	3.8	3.6

k = 4

P^* \ n_0	2	3	4	5	6	7	8	9	10	15	20	25	30	∞
.75	6.9	3.4	2.7	2.5	2.3	2.3	2.2	2.2	2.13	2.06	2.05	1.99	1.98	1.92
.80	8.8	3.8	3.0	2.8	2.6	2.5	2.4	2.4	2.33	2.26	2.24	2.19	2.18	2.09
.85	11.9	4.5	3.4	3.1	2.9	2.8	2.7	2.6	2.59	2.50	2.46	2.39	2.40	2.29
.90	18.2	5.7	4.0	3.6	3.3	3.2	3.1	3.0	2.95	2.80	2.77	2.67	2.70	2.58
.95	35	8.0	5.2	4.5	4.0	3.8	3.6	3.6	3.45	3.31	3.21	3.13	3.15	3.04
.975	70	11	6.8	5.5	4.7	4.5	4.3	4.1	4.00	3.78	3.66	3.55	3.53	3.41
.99	170	18	9	7	6	5.3	5.1	4.9	4.6	4.2	4.1	4.1	3.9	3.8

Cont. Table 5

k = 5

p^* \ n_0	2	3	4	5	6	7	8	9	10	15	20	25	30	∞
.75	9.2	4.0	3.1	2.8	2.6	2.5	2.4	2.4	2.35	2.28	2.26	2.22	2.20	2.13
.80	11.7	4.5	3.4	3.1	2.9	2.8	2.7	2.6	2.57	2.49	2.44	2.39	2.37	2.30
.85	15.8	5.3	3.9	3.4	3.2	3.0	2.9	2.9	2.83	2.72	2.67	2.59	2.59	2.51
.90	23.9	6.6	4.5	4.0	3.6	3.5	3.3	3.2	3.18	3.01	2.95	2.86	2.86	2.78
.95	47	9.4	5.9	4.8	4.3	4.1	3.9	3.8	3.69	3.51	3.42	3.30	3.30	3.18
.975	95	14	7.4	5.9	5.1	4.8	4.5	4.4	4.24	3.96	3.83	3.72	3.72	3.57
.99	230	21	10	8	7	5.6	5.3	5.0	4.9	4.4	4.3	4.3	4.2	3.9

k = 6

p^* \ n_0	2	3	4	5	6	7	8	9	10	15	20	25	30	∞
.75	11.6	4.4	3.4	3.1	2.9	2.7	2.6	2.6	2.54	2.47	2.43	2.37	2.37	2.28
.80	14.8	5.0	3.7	3.3	3.1	3.0	2.8	2.8	2.75	2.64	2.61	2.54	2.54	2.45
.85	20.1	5.9	4.2	3.7	3.4	3.2	3.1	3.1	3.01	2.87	2.83	2.74	2.75	2.64
.90	31	7.3	4.9	4.3	3.8	3.6	3.5	3.4	3.33	3.17	3.11	3.01	3.03	2.92
.95	62	10.5	6.3	5.1	4.5	4.3	4.1	3.9	3.85	3.67	3.56	3.45	3.43	3.30
.975	125	15	7.9	6.2	5.5	5.0	4.7	4.5	4.38	4.14	3.96	3.88	3.80	3.65
.99	315	22	11	8	7	5.7	5.5	5.2	5.0	4.5	4.4	4.4	4.3	4.0

k = 7

p^* \ n_0	2	3	4	5	6	7	8	9	10	15	20	25	30	∞
.75	13.7	4.9	3.6	3.2	3.0	2.9	2.8	2.8	2.68	2.59	2.55	2.49	2.49	2.41
.80	17.6	5.5	4.0	3.5	3.3	3.1	3.0	3.0	2.89	2.77	2.73	2.65	2.65	2.57
.85	19.8	6.5	4.5	3.9	3.6	3.4	3.3	3.2	3.15	2.98	2.95	2.85	2.85	2.75
.90	36	8.0	5.2	4.5	4.0	3.8	3.6	3.6	3.45	3.28	3.21	3.12	3.12	3.02
.95	73	12	6.8	5.4	4.7	4.5	4.3	4.1	4.00	3.77	3.68	3.55	3.53	3.39
.975	143	16	8.4	6.6	5.6	5.1	4.9	4.6	4.54	4.23	4.08	3.99	3.91	3.73
.99	350	25	11	9	7	6	6	5.3	5.3	4.7	4.5	4.4	4.3	4.2

Cont. Table 5

$k = 8$

p^* \ n_0	2	3	4	5	6	7	8	9	10	15	20	25	30	∞
.75	15.9	5.2	3.9	3.4	3.2	3.0	2.9	2.9	2.82	2.70	2.66	2.58	2.58	2.50
.80	20.3	6.0	4.3	3.7	3.4	3.3	3.1	3.1	3.02	2.87	2.84	2.75	2.75	2.65
.85	27.4	7.0	4.8	4.1	3.7	3.5	3.4	3.3	3.25	3.08	3.05	2.95	2.94	2.85
.90	42	8.6	5.5	4.7	4.2	4.0	3.8	3.7	3.57	3.39	3.32	3.22	3.22	3.10
.95	85	12	7.1	5.7	4.9	4.6	4.4	4.3	4.10	3.85	3.75	3.64	3.61	3.47
.975	175	17	8.7	7.0	5.9	5.2	5.0	4.8	4.62	4.30	4.18	4.16	4.01	3.79
.99	410	27	12	9	7	6	6	5.5	5.5	4.8	4.5	4.5	4.5	4.2

$k = 9$

p^* \ n_0	2	3	4	5	6	7	8	9	10	15	20	25	30	∞
.75	18.2	5.6	4.1	3.6	3.3	3.2	3.0	3.0	2.92	2.79	2.77	2.67	2.67	2.59
.80	22.9	6.4	4.5	3.9	3.6	3.4	3.3	3.2	3.12	2.97	2.93	2.83	2.83	2.73
.85	31.2	7.5	5.0	4.3	3.9	3.7	3.5	3.4	3.34	3.18	3.14	3.03	3.03	2.93
.90	47	9.2	5.8	4.8	4.3	4.1	3.9	3.8	3.67	3.48	3.41	3.31	3.29	3.16
.95	94	13	7.4	5.9	5.1	4.7	4.5	4.4	4.20	3.93	3.84	3.74	3.68	3.53
.975	200	18	9.1	7.2	6.0	5.3	5.1	4.9	4.74	4.36	4.27	4.16	4.11	3.87
.99	500	30	13	9	8	6	6	5.6	5.5	4.9	4.6	4.6	4.5	4.3

$k = 10$

p^* \ n_0	2	3	4	5	6	7	8	9	10	15	20	25	30	∞
.75	20.3	6.0	4.3	3.7	3.4	3.3	3.1	3.1	3.03	2.88	2.84	2.76	2.76	2.65
.80	25.7	6.8	4.7	4.0	3.7	3.5	3.4	3.3	3.20	3.06	3.01	2.91	2.91	2.81
.85	35	7.9	5.2	4.4	4.0	3.8	3.6	3.5	3.43	3.28	3.20	3.12	3.12	2.99
.90	52	9.7	6.0	5.0	4.5	4.2	4.0	3.9	3.75	3.56	3.47	3.39	3.37	3.22
.95	109	14.0	7.6	6.0	5.3	4.9	4.6	4.5	4.27	4.01	3.90	3.82	3.74	3 58
.975	220	19	9.6	7.4	6.2	5.4	5.2	5.0	4.8	4.42	4.29	4.21	4.12	3.92
.99	520	30	13	9.3	8	6.3	6.0	5.7	5.7	4.9	4.7	4.7	4.6	4.3

Cont. Table 5

k = 11

p^* \ n_0	2	3	4	5	6	7	8	9	10	15	20	25	30	∞
.75	22.5	6.3	4.4	3.8	3.5	3.4	3.1	3.1	3.10	2.96	2.90	2.82	2.82	2.72
.80	28.9	7.1	4.9	4.1	3.8	3.6	3.4	3.3	3.28	3.13	3.06	2.98	2.98	2.88
.85	39	8.2	5.4	4.5	4.1	3.9	3.6	3.5	3.50	3.34	3.25	3.17	3.17	3.05
.90	58	10.2	6.2	5.1	4.6	4.3	4.0	3.9	3.83	3.61	3.53	3.45	3.42	3.28
.95	123	14.5	7.9	6.2	5.4	5.0	4.6	4.5	4.36	4.07	3.94	3.89	3.81	3.64
.975	240	20	9.8	7.6	6.3	5.5	5.2	5.0	4.90	4.47	4.38	4.29	4.21	3.97
.99	580	31	13	10	8	6	6	5.7	5.7	4.9	4.7	4.7	4.6	4.4

k = 12

p^* \ n_0	2	3	4	5	6	7	8	9	10	15	20	25	30	∞
.75	24.7	6.5	4.6	4.0	3.6	3.5	3.3	3.2	3.17	3.02	2.97	2.88	2.88	2.78
.80	31.6	7.4	5.0	4.3	3.9	3.7	3.5	3.4	3.35	3.18	3.13	3.05	3.05	2.93
.85	43	8.6	5.6	4.7	4.2	4.0	3.8	3.7	3.58	3.42	3.32	3.25	3.24	3.11
.90	66	10.5	6.4	5.2	4.7	4.4	4.2	4.0	3.89	3.68	3.60	3.52	3.48	3.33
.95	135	15.1	8.1	6.3	5.5	5.1	4.8	4.6	4.42	4.12	4.02	3.94	3.88	3.69
.975	260	21	10.1	7.7	6.4	5.6	5.3	5.1	4.98	4.51	4.45	4.33	4.25	4.05
.99	620	32	13	10	8	7	6	5.8	5.8	5.1	4.8	4.7	4.7	4.5

k = 13

p^* \ n_0	2	3	4	5	6	7	8	9	10	15	20	25	30	∞
.75	27.1	6.8	4.7	4.1	3.7	3.5	3.4	3.3	3.23	3.08	3.02	2.94	2.93	2.84
.80	34.3	7.7	5.1	4.4	4.0	3.7	3.6	3.5	3.41	3.26	3.18	3.10	3.09	2.98
.85	46	8.9	5.7	4.8	4.3	4.0	3.9	3.8	3.65	3.47	3.38	3.31	3.27	3.16
.90	72	11.1	6.6	5.4	4.8	4.5	4.2	4.1	3.97	3.73	3.65	3.57	3.52	3.38
.95	150	15.7	8.2	6.4	5.6	5.1	4.8	4.6	4.50	4.18	4.07	4.00	3.93	3.75
.975	285	22	10.4	7.9	6.5	5.7	5.4	5.2	5.06	4.58	4.50	4.37	4.30	4.08
.99	660	35	14	10	8	7	6	5.9	5.9	5.1	4.3	4.8	4.8	4.5

Cont. Table 5

k = 14

p^* \ n_0	2	3	4	5	6	7	8	9	10	15	20	25	30	∞
.75	29.7	7.1	4.9	4.2	3.8	3.6	3.5	3.4	3.29	3.13	3.07	2.99	2.98	2.88
.80	32	8.0	5.3	4.5	4.1	3.8	3.7	3.6	3.47	3.31	3.22	3.15	3.14	3.03
.85	50	9.3	5.9	4.9	4.4	4.1	3.9	3.8	3.71	3.52	3.43	3.36	3.32	3.19
.90	81	11.5	6.8	5.5	4.8	4.5	4.3	4.2	4.02	3.78	3.70	3.61	3.57	3.43
.95	170	16	8.4	6.6	5.7	5.1	4.9	4.7	4.57	4.22	4.13	4.05	3.96	3.78
.975	310	22	10.7	8.0	6.7	5.9	5.5	[illegible]	5.12	4.65	4.57	4.42	4.33	4.09
.99	700	36	14	10	8	7	6	6.0	5.9	5.2	4.9	4.8	4.8	4.5

k = 15

p^* \ n_0	2	3	4	5	6	7	8	9	10	15	20	25	30	∞
.75	31.7	7.3	5.0	4.3	3.9	3.7	3.5	3.4	3.33	3.18	3.11	3.04	3.03	2.92
.80	41	8.2	5.4	4.6	4.1	[illegible]	3.7	3.6	3.52	3.36	3.26	3.20	3.19	3.06
.85	55	9.6	6.0	5.0	4.5	4.2	4.0	3.9	3.76	3.57	3.47	3.40	3.37	3.23
.90	86	12.0	7.0	5.6	4.9	4.6	4.4	4.2	4.08	3.83	3.73	3.67	3.60	3.46
.95	175	17	8.7	6.7	5.8	5.2	5.0	4.8	4.61	4.26	4.18	4.09	4.02	3.81
.975	340	23	10.8	8.2	6.8	5.9	5.6	5.3	5.19	4.68	4.60	4.44	4.39	4.13
.99		37	15	10	8	7	6	6.0	6.0	5.2	4.9	4.8	4.8	4.5

k = 16

p^* \ n_0	2	3	4	5	6	7	8	9	10	15	20	25	30	∞
.75	34.1	7.6	5.1	4.3	3.9	3.7	3.6	3.5	3.39	3.22	3.16	3.08	3.08	2.96
.80	44	8.6	5.5	4.7	4.2	4.0	3.8	3.7	3.58	3.41	3.31	3.24	3.23	3.10
.85	59	10.0	6.2	5.1	4.5	4.3	4.1	3.9	3.82	3.60	3.51	3.45	3.41	3.27
.90	91	12.6	7.1	5.7	5.0	4.7	4.4	4.3	4.14	3.87	3.77	3.71	3.65	3.48
.95	190	17	8.9	6.9	5.9	5.3	5.0	4.8	4.67	4.31	4.18	4.12	4.07	3.84
.975	370	24	11.1	8.3	6.9	6.0	5.7	5.4	5.25	4.73	4.62	4.47	4.43	4.17
.99		38	15	10	8	7	7	6.0	6.0	5.2	4.9	4.9	4.9	4.5

Cont. Table 5

k = 17

p* \ n_0	2	3	4	5	6	7	8	9	10	15	20	25	30	∞
.75	36.6	7.8	5.2	4.4	4.0	3.8	3.6	3.5	3.44	3.27	3.19	3.13	3.12	3.00
.80	47	8.8	5.6	4.7	4.3	4.0	3.8	3.7	3.63	3.46	3.35	3.29	3.26	3.13
.85	64	10.3	6.3	5.2	4.6	4.3	4.1	4.0	3.86	3.64	3.54	3.49	3.44	3.30
.90	98	12.9	7.3	5.8	5.1	4.7	4.5	4.3	4.19	3.94	3.79	3.75	3.69	3.52
.95	200	18	9.1	7.0	6.0	5.4	5.1	4.9	4.73	4.35	4.25	4.15	4.10	3.88
.975	390	25	11.4	8.5	7.0	6.1	5.8	5.4	5.32	4.77	4.64	4.51	4.46	4.21
.99		39	16	10	8	7	7	6.1	6.1	5.3	5.0	4.9	4.9	4.6

k = 18

p* \ n_0	2	3	4	5	6	7	8	9	10	15	20	25	30	∞
.75	38	8.0	5.3	4.5	4.1	3.8	3.7	3.6	3.48	3.31	3.23	3.17	3.15	3.03
.80	49	9.1	5.8	4.8	4.3	4.1	3.9	3.8	3.68	3.50	3.38	3.33	3.30	3.17
.85	67	10.5	6.4	5.2	4.7	4.4	4.2	4.0	3.91	3.69	3.58	3.52	3.48	3.33
.90	102	13.1	7.4	5.8	5.1	4.8	4.5	4.4	4.24	3.98	3.83	3.79	3.73	3.57
.95	210	18	9.3	7.1	6.1	5.4	5.1	5.0	4.78	4.40	4.29	4.18	4.13	3.91
.975	410	25	11.6	8.6	7.1	6.2	5.8	5.5	5.35	4.81	4.67	4.53	4.47	4.22
.99		39	16	11	8	7	7	6.1	6.1	5.3	5.0	5.0	4.9	4.6

k = 19

p* \ n_0	2	3	4	5	6	7	8	9	10	15	20	25	30	∞
.75	41	8.2	5.4	4.6	4.1	3.9	3.7	3.6	3.53	3.35	3.27	3.20	3.19	3.06
.80	52	9.3	5.9	4.9	4.4	4.1	3.9	3.8	3.72	3.53	3.43	3.37	3.33	3.20
.85	70	10.8	6.5	5.3	4.7	4.4	4.2	4.1	3.95	3.73	3.62	3.56	3.51	3.36
.90	107	13.4	7.5	5.9	5.2	4.8	4.6	4.4	4.28	4.02	3.88	3.82	3.77	3.58
.95	215	19	9.6	7.2	6.1	5.5	5.2	5.0	4.83	4.42	4.33	4.22	4.16	3.92
.975	420	26	11.9	8.7	7.1	6.3	5.9	5.5	5.38	4.83	4.70	4.57	4.51	4.24
.99		40	16	11	9	8	7	6.2	6.1	5.4	5.1	5.0	5.0	4.6

Cont. Table 5

k = 20

p^{*} \ n_0	2	3	4	5	6	7	8	9	10	15	20	25	30	∞
.75	43	8.5	5.5	4.6	4.2	3.9	3.8	3.7	3.57	3.41	3.30	3.23	3.22	3.09
.80	54	9.5	6.0	5.0	4.4	4.2	4.0	3.9	3.75	3.57	3.46	3.40	3.37	3.22
.85	75	11.1	6.7	5.4	4.8	4.5	4.3	4.1	3.99	3.77	3.65	3.58	3.53	3.38
.90	112	13.9	7.7	6.0	5.3	4.9	4.6	4.5	4.32	4.05	3.90	3.85	3.79	3.61
.95	225	19	9.7	7.3	6.2	5.5	5.2	5.0	4.85	4.47	4.35	4.25	4.18	3.94
.975	440	27	12.0	8.8	7.2	6.3	5.9	5.6	5.43	4.87	4.73	4.58	4.53	4.26
.99		40	17	11	9	8	7	6.2	6.2	5.4	5.1	5.0	5.0	4.6

k = 21

p^{*} \ n_0	2	3	4	5	6	7	8	9	10	15	20	25	30	∞
.75	45	8.7	5.6	4.7	4.2	4.0	3.8	3.7	3.62	3.44	3.34	3.27	3.24	3.12
.80	57	9.8	6.1	5.0	4.5	4.2	4.0	3.9	3.80	3.60	3.50	3.43	3.39	3.25
.85	79	11.5	6.8	5.4	4.8	4.5	4.3	4.2	4.03	3.81	3.69	3.62	3.57	3.41
.90	119	14.2	7.8	6.1	5.3	4.9	4.7	4.5	4.37	4.08	3.95	3.87	3.82	3.63
.95	235	20	9.8	7.4	6.2	5.6	5.3	5.1	4.90	4.49	4.39	4.28	4.20	3.96
.975	470	27	12	8.9	7.3	6.4	6.0	5.6	5.45	4.89	4.76	4.59	4.56	4.27
.99		40	17	11	9	8	7	6.4	6.2	5.4	5.1	5.1	5.0	4.6

k = 22

p^{*} \ n_0	2	3	4	5	6	7	8	9	10	15	20	25	30	∞
.75	47	9.0	5.7	4.8	4.3	4.0	3.9	3.8	3.65	3.47	3.37	3.30	3.27	3.14
.80	61	10.1	6.2	5.1	4.6	4.3	4.1	4.0	3.83	3.63	3.53	3.46	3.42	3.27
.85	83	11.8	6.9	5.5	4.9	4.6	4.3	4.2	4.07	3.84	3.72	3.65	3.59	3.44
.90	127	14.6	7.9	6.1	5.4	5.0	4.7	4.5	4.39	4.11	3.98	3.90	3.84	3.65
.95	245	20	10.0	7.4	6.3	5.6	5.3	5.1	4.93	4.52	4.30	4.31	4.22	3.97
.975	480	28	12	9.0	7.4	6.4	6.0	5.6	5.48	4.93	4.77	4.63	4.60	4.30
.99		41	17	11	9	8	7	6.4	6.3	5.4	5.1	5.1	5.0	4.7

Cont. Table 5

k = 23

p^* \ n_0	2	3	4	5	6	7	8	9	10	15	20	25	30	∞
.75	49	9.2	5.8	4.8	4.3	4.1	3.9	3.8	3.69	3.50	3.40	3.33	3.30	3.17
.80	64	10.3	6.3	5.1	4.6	4.3	4.1	4.0	3.87	3.66	3.56	3.49	3.44	3.30
.85	86	12.1	7.0	5.5	4.9	4.6	4.4	4.2	4.10	3.86	3.74	3.68	3.62	3.47
.90	134	14.9	8.0	6.2	5.5	5.0	4.8	4.5	4.42	4.13	4.00	3.93	3.87	3.68
.95	255	21	10.1	7.5	6.4	5.7	5.4	5.2	4.98	4.53	4.42	4.34	4.24	4.00
.975	500	28	13	9.1	7.4	6.5	6.1	5.7	5.52	4.96	4.79	4.66	4.62	4.32
.99		41	17	12	9	8	7	6.4	6.3	5.5	5.1	5	5.0	4.7

k = 24

p^* \ n_0	2	3	4	5	6	7	8	9	10	15	20	25	30	∞
.75	52	9.3	5.9	4.9	4.4	4.1	3.9	3.8	3.72	3.52	3.43	3.37	3.33	3.19
.80	67	10.4	6.4	5.2	4.7	4.4	4.2	4.0	3.90	3.69	3.58	3.52	3.47	3.32
.85	90	12.4	7.1	5.6	5.0	4.7	4.4	4.3	4.13	3.90	3.77	3.71	3.65	3.48
.90	141	15.3	8.2	6.3	5.5	5.1	4.8	4.6	4.46	4.16	4.02	3.95	3.90	3.69
.95	265	21	10.2	7.6	6.5	5.8	5.4	5.2	4.99	4.57	4.43	4.35	4.27	4.02
.975	520	29	13	9.1	7.5	6.5	6.1	5.7	5.54	4.98	4.80	4.67	4.63	4.33
.99			17	12	9	8	7	6.4	6.3	5.5	5.2	5.1	5.0	4.7

k = 25

p^* \ n_0	2	3	4	5	6	7	8	9	10	15	20	25	30	∞
.75	54	9.5	5.9	4.9	4.4	4.2	4.0	3.9	3.74	3.55	3.45	3.39	3.36	3.22
.80	70	10.8	6.5	5.2	4.7	4.4	4.2	4.0	3.93	3.73	3.61	3.55	3.49	3.34
.85	94	12.7	7.2	5.7	5.0	4.7	4.5	4.3	4.17	3.93	3.79	3.74	3.68	3.50
.90	146	15.6	8.2	6.3	5.6	5.0	4.8	4.7	4.49	4.18	4.04	3.98	3.92	3.72
.95	285	22	10.3	7.7	6.5	5.8	5.4	5.2	5.06	4.61	4.46	4.37	4.29	4.03
.975	560	30	13	9.2	7.6	6.5	6.2	5.7	5.65	5.01	4.82	4.71	4.66	4.34
.99			18	12	9	8	7.0	6.5	6.4	5.5	5.2	5.2	5.1	4.7

6. CONCLUSIONS AND RECOMMENDATIONS

In this paper we have given tables needed to implement new procedures for ranking a set of $k(\geq 2)$ independent normal populations with unknown means and variances. The procedure is based on two-stage sampling from each population, whereby the same initial sample size n_0 (≥ 2) is taken from each population. The examples provided illustrate the wide applicability of this procedure, point out the various steps involved in carrying out the procedures, and indicate proper use of the tables. The tables are believed to be accurate within error of a Monte Carlo procedure using 10,000 samples. The tables should be helpful to any experimenter who wishes to design and carry out his experiment with the goal of correctly ranking several populations with a high probability of being correct. Ranking procedures in an analysis of variance setting are receiving increasing emphasis in the field of ranking and selection. Procedures for analysis of the data of example 2 supplemented with data from other locations as well are currently under development.

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